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The Correlation Function in Wilson's Theory of Phase Transitions: Above and Below the Transition

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The correlation function and the correlation length are discussed in the theoretical framework of the Wilson–Feynman diagram expansion for small $\epsilon = 4 - d$. It is shown explicitly that to order ϵ^2 the scaling relation $\gamma = \nu(2 - \eta)$ is satisfied and that the correlation function is a homogeneous function of k and ξ. The explicit form of the scaled correlation function is exhibited.

KEY WORDS: Correlation function; ε-expansion; ordered state; phase transition.

1. INTRODUCTION

One of the most important recent developments in the theory of critical phenomena is a perturbation expansion for the critical exponents in powers of $\epsilon = 4 - d$, where d is the number of dimensions of the space. Invented by Wilson,[(1)] it was first used to calculate the critical exponents γ and η to second order in ϵ; it was then extended by Brezin *et al.*[(2,3)] to situations in which order is present in the system.

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Besides calculating β and δ to the same order in ϵ, they showed that up to this order the system obeys a homogeneous equation of state. Since it was pointed out by Griffiths[4] that for such systems $\alpha = 2 - (\gamma + 2\beta)$, α does not have to be calculated explicitly and the critical exponent left to be calculated is ν—the one corresponding to the correlation length.

The value of ν does not, of course, follow from the equation of state. It does, however, follow if the correlation function is a homogeneous function of k and ξ, where ξ is the correlation length. In that case ν can be determined from the scaling relation[5]

$$\gamma = \nu(2 - \eta) \tag{1}$$

once γ and η are known.

It has been argued[6] that this homogeneity property follows from general renormalization group arguments together with the fact that we are dealing with a one-parameter theory. We feel that such arguments, though quite probably valid, do not make the calculation of the correlation function unnecessary. The reason is manifold. First, the equations in Ref. 6 and their solutions are discussed only in the disordered phase. Second, the equivalence of the Wilson–Feynman graph expansion and the method of Ref. 6 has not been demonstrated. They do agree on the value of η to order ϵ^2, but whether they lead to the same correlation function or to the same equation of state is not yet clear. Finally, we feel that the strength of very general and abstract theorems in mathematical physics stems from the constructive calculational procedures attached to them. Such procedures do, in fact, make their appearance in the context of the Callan–Symanzik[7] renormalization group equations (see, e.g., Ref. 8). The results of the general theory should now be anchored in these procedures.

Similarly, the Wilson–Feynman graph expansion is a well-defined method in which exponents can be calculated as an asymptotic series in $\epsilon = 4 - d$. It is within this context that we calculate ν both above and below the transition to order ϵ^2. We show that the scaling relation (1) is satisfied and that to the same order the correlation function is homogeneous.

Above the transition temperature the calculation is carried out with an order-parameter field possessing an arbitrary number n of components. However, below the transition our calculation is restricted to $n = 1$ in order to avoid additional complications arising from the appearance of Goldstone bosons (see Ref. 3). There is no reason to expect any qualitative differences.

2. REVIEW OF NOTATION

The formulation of the phase transition problem in terms of a functional average, to be used here, has been already described by a number of authors (see, e.g., Refs. 1, 3, and 9). Therefore we shall limit ourselves to some brief notes.

The Hamiltonian of the Ising-like system (for the generalization to the n-vector model see Ref. 3) can be divided into two parts as follows[2]:

$$H_0/kT = \int d^dx\{\tfrac{1}{2}rs^2(\mathbf{x}) + \tfrac{1}{2}[\nabla s(\mathbf{x})]^2\} \tag{2}$$

$$\begin{aligned} H_1/kT = \int d^dx[u_0s^4(\mathbf{x}) + 4u_0ms^3(\mathbf{x}) + \tfrac{1}{2}(\delta r_1)\, s^2(\mathbf{x}) \\ + (r_0 + 4u_0m^2)\, ms(\mathbf{x}) + \tfrac{1}{2}r_0m^2 + u_0m^4] \end{aligned} \tag{3}$$

where H_1 is used as a perturbation and the new symbols are

$$\delta r_1 = r_0 - r + 12u_0m^2 \tag{4}$$

with r the reciprocal of the susceptibility and m the uniform part of the order-parameter field.

In complete analogy to the treatment of many-body problems, the inverse correlation function satisfies a Dyson equation[9]:

$$g^{-1}(k; r) = g_0^{-1}(k; r) - M(k; r) \tag{5}$$

with a self-energy part $M(k; r)$ and a free propagator

$$g_0(k; r) = (k^2 + r)^{-1} \tag{6}$$

Since the susceptibility χ is given in terms of the full correlation function

$$\chi^{-1} = r = g^{-1}(k = 0; r) \tag{7}$$

the "mass" renormalization implied by the choice (6) for $g_0(k; r)$ is expressed as

$$M(k = 0; r) = 0 \tag{8}$$

meaning that there are no self-energy insertions for $k = 0$.

The expansion of $M(k; r)$ for $n = 1$ in terms of δr_1 (diagramatically represented by $\times$) and u_0 (represented by a dot) to second order, in the explicit dependence on δr_1 and u_0, includes the terms (see Fig. 1)

$$\begin{aligned} M(k; r) = -\delta r_1 - 12u_0D_1(r) + 12u_0\delta r_1D_2(r) + 144u_0{}^2D_2(r)\, D_1(r) \\ + 288u_0{}^2\overline{m}^2D_2(k; r) + 96u_0{}^2D_3(k; r) \end{aligned} \tag{9}$$

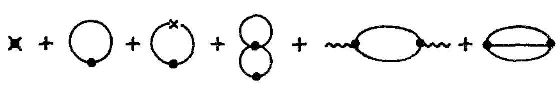

Fig. 1. Diagrams for the self-energy which include two vertices or less.

The solid line in Fig. 1 represents, diagramatically, a free propagator $g_0(k; r)$, and the wavy line represents the equilibrium value of the order parameter $\bar{m}$. A more detailed discussion of $\bar{m}$ will be presented in Section 3. The symbols $D_i(k; r)$ stand for the integrals

$$D_1(r) = \int d^d q(\mathbf{q}^2 + r)^{-1} \tag{10}$$

$$D_2(k; r) = \int d^d q(\mathbf{q}^2 + r)^{-1}[(\mathbf{q} + \mathbf{k})^2 + r]^{-1}, \qquad D_2(k = 0; r) \equiv D_2(r) \tag{11}$$

$$D_3(k; r) = \int d^d q\, d^d p(\mathbf{q}^2 + r)^{-1}(\mathbf{p}^2 + r)^{-1}[(\mathbf{q} + \mathbf{p} + \mathbf{k})^2 + r]^{-1}; \qquad D_3(k = 0; r) \equiv D_3(r) \tag{12}$$

3. THE ORDER PARAMETER

The value of the order parameter has to be known if we want to evaluate the contributions of the various terms of the expansion. In (3) we have introduced m as a uniform part of the field, which means it must be treated as an extra functional variable to be integrated over. However, such an integration can be avoided since the equilibrium value of m minimizes the free energy of the system, which is canonical with respect to m. We proceed as follows.

Extracting this particular integration from the expression of the partition function

$$Z = \int_{-\infty}^{+\infty} dm\, Z(m) \tag{13}$$

where

$$Z(m) = \exp[-V(\tfrac{1}{2}r_0 m^2 + u_0 m^4)] \int \mathscr{D}\{s\} \exp[-H'(m)/kT] \tag{14}$$

we calculate the free energy as a logarithm of the partition function taken in the infinite-volume limit:

$$F = -kT \lim_{V\to\infty} (1/V) \ln Z(\bar{m}) \tag{15}$$

That means, that $\bar{m}$, the equilibrium value of order parameter, is a minimum point (which is very sharp when $V \to \infty$) of a free energy as a function of m. Therefore it is calculated as a solution of the following equation:

$$(\partial/\partial m) \ln Z(m) = 0 \tag{16}$$

This procedure is, of course, completely equivalent[9] to the one used in Refs. 2 and 3. Nevertheless we reproduce in some detail the calculation of $\bar{m}$, since its explicit value, not pointed out in Refs. 2 and 3, will be needed in what follows.

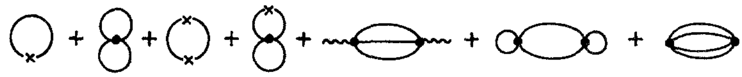

Fig. 2. Diagrams for the free energy which include two vertices or less.

The free energy of the system ($n = 1$) can be expanded using the Wilson–Feynman graph technique. Up to the second order in δr_1 and u_0 the terms in the expansion can be given as (see Fig. 2)

$$F(m) = F_0 - \tfrac{1}{2}r_0 m^2 - u_0 m^4 - \tfrac{1}{2}\delta r_1 D_1(r) - 3u_0 D_1{}^2(r) + (\tfrac{1}{2}\delta r_1)^2 D_2(r) + 12u_0(\tfrac{1}{2}\delta r_1) D_2(r) D_1(r) + 48u_0{}^2 m^2 D_3(r) + 36u_0{}^2 D_2(r) D_1{}^2(r) + 12u_0{}^2 D_4(r) \tag{17}$$

with F_0 an "unperturbed" free energy. The explicit form of $D_4(r)$ will not be needed.

Above T_c the only solution of Eq. (16) is $\bar{m} = 0$. Below T_c, after elimination of the trivial solution $\bar{m} = 0$, Eq. (16) takes the form

$$r - 8u_0\bar{m}^2 + 288u_0{}^2\bar{m}^2 D_2(k = 0; r) + \text{higher terms} = 0 \tag{18}$$

where the renormalization expressed by Eq. (8) was used. The value of the coupling constant $u_0 = (2\pi^2/9)\,\epsilon$ has to be chosen in order to produce the proper critical behavior of $\bar{m}$. The asymptotic solution of Eq. (18) can be obtained in the following way. Expanding the equilibrium value of the order parameter in powers of $\ln r$

$$\bar{m}^2 = \Gamma r^{1-\Delta} \approx \Gamma r(1 - \Delta \ln r + \cdots) \tag{19}$$

and evaluating $D_2(k = 0; r)$ to give

$$D_2(k = 0; r) \approx (4\pi)^{-2}\,[-\ln r + \ln \Lambda^2 + (2r/\Lambda^2) - 1] \tag{20}$$

(Λ is a momentum cutoff), we insert these expressions into Eq. (18) and find the latter expanded in powers of $r, r^2, \ldots, r \ln r, r \ln^2 r, \ldots$, and so on.

Obviously, the equation holds for a finite interval of values of r if all coefficients in this expansion equal zero. Thus equating the coefficient of r to zero, we find, to lowest order in ϵ, $\Gamma = 1/8u_0$. The same procedure with the coefficient of $r \ln r$ leads to $\Delta = \epsilon/2$. The equilibrium value of m follows:

$$\bar{m}^2 \approx (3/4\pi)^2\,(1/\epsilon)\, r(1 - \tfrac{1}{2}\epsilon \ln r + \cdots) \tag{21}$$

4. REMARKS CONCERNING THE ANALYSIS OF DIAGRAMS

As a result of the mass renormalization, Eq. (8), we find that $\delta r_1 \sim \epsilon$. To see this, one inserts Eq. (9) into (8) and replaces u_0 by its special value, which

Fig. 3

Fig. 4

is of order ϵ. Thus in order to perform a calculation to a given order in ϵ, we have to keep all diagrams which include no more vertices than the desired order. This applies to vertices of both kinds, i.e., two- and four-point vertices. There is one exception to this rule, namely when order-parameter "legs" appear in diagrams below the transition. Since the order parameter appears as a factor in a three-point interaction, it always enters diagrams in pairs, that is, in the combination $u_0{}^2\bar{m}^2$. Furthermore, according to Eq. (21),[2,3] $u_0\bar{m}^2$ is of order unity. Thus, two vertices with $\bar{m}$ "legs" raise the order of a diagram by only one power of ϵ. For example, the diagram in Fig. 3 contributes to first order in ϵ.

As a consequence, the only diagram of M that has to be considered up to order ϵ^2 above T_c, which is also the only one without $\bar{m}$ "legs" below T_c, is the one shown in Fig. 4.

On the other hand, quite a few diagrams which include $\bar{m}$ have to be considered up to this order in ϵ. They are shown in Fig. 5. These diagrams will be considered in detail below.

Finally, such diagrams as shown in Fig. 6, which are "dangerous" since they propagate the same momentum inside of a line and thus become very divergent on integration, do not enter. They all cancel, to all orders, as a special case of renormalization of the propagator at $k = 0$, Eq. (8).

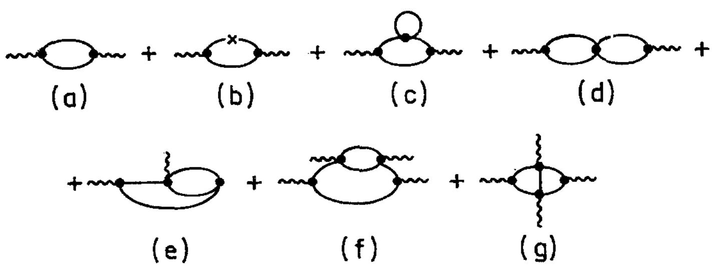

Fig. 5

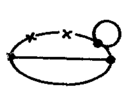

Fig. 6

5. CORRELATION LENGTH

We define the inverse of the correlation length as an imaginary pole of $g(k; r)$,(8,10,12) since it gives the exponential decay range of the spatial correlation function, namely

$$g^{-1}(k = i\xi^{-1}; r) = 0 \tag{22}$$

Thus in the interval from $k^2 = -\xi^{-2}$, where $g^{-1}(k^2 = -\xi^{-2}; r) = 0$, to $k^2 = 0$, with $g^{-1}(k^2 = 0; r) = r$, $g^{-1}(k^2; r)$ is an increasing function of k^2. Since at $r \to 0$, $\xi^{-2} \to 0$, it can be approximated in this interval by the straight line, with an accuracy dependent on $T - T_c$. Then it follows that

$$\xi^{-2} \approx \alpha / r \tag{23}$$

with α as a derivative of $g^{-1}(k^2; r)$ with respect to k^2 at $k^2 = 0$.

It is easy to see that the definition (22), up to a constant factor, coincides with the one given in Ref. 12, where ξ^2 is defined as an effective quadratic extent of the correlation function:

$$\xi^2 = \chi^{-1} \left. \frac{\partial^2 g(k; r)}{\partial k^2} \right|_{k=0}$$

which after some simple transformations takes the form

$$\xi^{-2} = \frac{2}{r} \left. \frac{\partial g^{-1}(k^2; r)}{\partial (k^2)} \right|_{k^2=0} \tag{25}$$

We make use of the definition (22) and Eqs. (5) and (8) to obtain:

$$g^{-1}(k = i\xi^{-1}; r) = 0 = -\xi^{-2} + r - [M(k = i\xi^{-1}; r) - M(k = 0; r)] \tag{26}$$

Since the critical behavior of $g(k; r)$ is known,

$$g^{-1}(k; r = 0) = A(\epsilon)\, k^{2-\eta} \tag{27}$$

another useful relation can be derived. With $g^{-1}(k = 0; r = 0) = 0$ we arrive at

$$g^{-1}(k; r = 0) = Ak^{2-\eta} = k^2 - [M(k; r = 0) - M(k = 0; r = 0)] \tag{28}$$

from which one usually determines η. Since $\overline{m} = 0$ at $T = T_c$, the only term in (28) which does not vanish to second order in ϵ is

$$96 u_0^2 [D_3(k; r = 0) - D_3(k = 0; r = 0)] \tag{29}$$

which corresponds to the diagram of Fig. 4 [eq. (12)]. Evaluation to lowest order in ϵ gives

$$D_3(k; 0) - D_3(0; 0) \approx (4\pi)^{-4} k^2 (\ln k - \ln \Lambda - \tfrac{5}{4}) \tag{30}$$

The asymptotic solution of η can be easily obtained by analogy to the method used in Section 3. We expand $k^{2-\eta}$ in the lhs of Eq. (28) in powers of $\ln k$, make use of (30), and equate the coefficients of k^2 and $k^2 \ln k$ on both sides of Eq. (28), to find

$$A(\epsilon) = 1 + O(\epsilon^2); \qquad \eta = \epsilon^2/54 \tag{31}$$

Finally, in order to derive the equation which will be used to determine ξ, we evaluate Eq. (28) at $k = i\xi^{-1} \cdot \mathbf{n}$ ($\mathbf{n}$ is a unit vector, and $k^{2-\eta}$ is considered as $k^2 \mid k \mid^{-\eta}$) and combine it with (26). The resulting equation reads

$$\begin{aligned} r - \xi^{-(2-\eta)} - \{[M(k = i\xi^{-1}; r) - M(k = 0; r)] \\ - [M(k = i\xi^{-1}; 0) - M(k = 0, 0)]\} = 0 \end{aligned} \tag{32}$$

All self-energy terms entering this equation can be separated into two parts, as follows:

$$r - \xi^{-(2-\eta)} - u_0^2 f_1(k = i\xi^{-1}; r) - u_0^2 \overline{m}^2 f_2(k = i\xi^{-1}; r) = 0 \tag{33}$$

where $u_0^2 f_1$ denotes $\overline{m}$-independent terms (contributes only one diagram, depicted in Fig. 4) and $u_0^2 \overline{m}^2 f_2$ denotes $\overline{m}$-dependent terms (contributing diagrams are shown in Fig. 5). These terms will be considered in detail in the following section.

6. CALCULATION OF THE CORRELATION LENGTH

The correlation length is assumed to have the form $\xi^{-1} \sim r^{1/(2-\rho)}$, leading to

$$\xi^{-(2-\eta)} = \xi_0(\epsilon)\, r^{1+(\rho-\eta)/2+O(\epsilon^3)}$$

Expanding $\xi^{-(2-\eta)}$ in powers of $\ln r$, we insert it into Eq. (33) and arrive at

$$r - \xi_0 r[1 + \tfrac{1}{2}(\rho - \eta) \ln r + \cdots] - u_0^2 f_1(k = i\xi^{-1}; r) - u_0^2 \overline{m}^2 f_2(k = i\xi^{-1}; r) = 0 \tag{34}$$

To lowest order in ϵ, $\xi_0 = 1$, and ρ is obtained by equating the coefficient of $r \ln r$ on the lhs of Eq. (34) to zero. In order to derive this coefficient, we have to extract the contributions of f_1 and f_2 to the term $r \ln r$. The analysis will be carried out both above and below the transition temperature.

6.1. The System Above the Transition

Since $\overline{m} = 0$ at $T > T_c$, only $f_1(k = i\xi^{-1}; r)$ is left in Eq. (34) and, as was already mentioned, the only contributing diagram is the one of Fig. 4

[for the corresponding integral see Eq. (12)]. Taking into account the symmetry factor, which is easily obtained in the case of the n-component field,

$$f_1(k; r) = 32(n + 2)[\Delta D_3(k; r) - \Delta D_3(k; 0)] \tag{35}$$

where we use the notation

$$\Delta F(k) = F(k) - F(0) \tag{36}$$

It is straightforward to show that the term $f_1(k = i\xi^{-1}; r)$ does not produce a factor $r \ln r$ (10). The argument is as follows: since f_1 enters Eq. (34) with a coefficient $u_0^2 \sim \epsilon^2$, f_1 has to be evaluated at $\epsilon = 0$. At $\epsilon = 0$ all the terms in f_1 are well defined at $r = \xi^{-1} = 0$ and, furthermore, the total derivative of f_1 with respect to r vanishes at $r = \xi^{-1} = 0$. Thus, up to order ϵ^2, $f_1(k = i\xi^{-1}; r)$ does not contribute a term of the form $r \ln r$, consequently, $\rho = \eta$ and Eq. (1) is satisfied at least to this order in ϵ.

6.2. The System Below the Transition

Below T_c the contribution of $f_1(k = i\xi^{-1}; r)$ is exactly the same as above, and therefore is of no interest. However, a new term, denoted

$$u_0^2 \bar{m}^2 f_2(k = i\xi^{-1}; r),$$

appears as a sum of contributions of the diagrams possessing order-parameter "legs". These contributions, after subtraction of their values at $k = 0$ [see Eq. (32)], lead to the explicit expression of f_2, up to second order in ϵ, at $n = 1$ as

$$\begin{aligned} f_2(k; r) = 288[&\Delta D_2(k; r) - 2\,\delta r_1\,\Delta D_2'(k; r) \\ &- 24u_0 D_1(r)\,\Delta D_2'(k; r) - 12u_0\,\Delta D_2^2(k; r) \\ &- 48u_0\,\Delta D_4(k; r) + 576u_0^2\bar{m}^2\,\Delta D_4'(k; r) + 576u_0^2\bar{m}^2\,\Delta D_5(k; r)] \end{aligned} \tag{37}$$

where exposition of various expansion terms follows the order of contributing diagrams depicted in Fig. 5. The new symbols are

$$D_2'(k; r) = \int d^dq(\mathbf{q}^2 + r)^{-2}[(\mathbf{q} + \mathbf{k})^2 + r]^{-1} \tag{38}$$

[a common part of diagrams (b) and (c)];

$$D_4(k; r) = \int d^dq\, d^dp(\mathbf{q}^2 + r)^{-1}[(\mathbf{q} + \mathbf{k})^2 + r]^{-1}(\mathbf{p}^2 + r)^{-1}[(\mathbf{q} + \mathbf{p})^2 + r]^{-1} \tag{39}$$

[diagram (e)];

$$D_4'(k; r) = \int d^dq\, d^dp(\mathbf{q}^2 + r)^{-2}[(\mathbf{q} + \mathbf{k})^2 + r]^{-1}(\mathbf{p}^2 + r)^{-1}[(\mathbf{q} + \mathbf{p})^2 + r]^{-1} \tag{40}$$

[diagram (f)];

$$D_5(k; r) = \int d^dq\, d^dp(\mathbf{q}^2 + r)^{-1}[(\mathbf{q} + \mathbf{k})^2 + r]^{-1}[(\mathbf{q} - \mathbf{p})^2 + r]^{-1} \times (\mathbf{p}^2 + r)^{-1}[(\mathbf{p} + \mathbf{k})^2 + r]^{-1} \tag{41}$$

[diagram (g)].

Since $u_0^2\overline{m}^2 \sim \epsilon r(1 - \frac{1}{2}\epsilon \ln r)$, f_2 has to be evaluated to first order in ϵ. However, only its first term—contribution of the first-order diagram (a)—has to be evaluated to this order, while the next ones, arising from the second-order diagrams (b)–(g), have to be evaluated only to lowest order in ϵ ($\epsilon = 0$).

To extract the ($\ln r$) dependence of these terms, we note the following. Every "bubblelike" part of the diagram at $\mathbf{q} = 0$, where $\mathbf{q}$ is the transfered momentum of the "bubble," produces a contribution to the term $\ln r$ [cf. Eq. (20)]. Among the second-order diagrams considered such "bubbles" appear in diagrams (d), (e), and (f). These diagrams can be divided into two classes [see appendix, Eqs. (A.4)–(A.6)]—those that are ($\ln r$) dependent [since they include $D_2(k = 0; r) = D_2(r)$] and those that are ($\ln r$) independent, the latter being neglected in the analysis of this section.

Since the $D_2(k = 0; r)$ part of diagram (f) together with δr_1 and the $u_0 D_1(r)$ insertions of diagrams (b) and (c) form, to first order in ϵ, the renormalized self-energy $M(k = 0) = 0$, the sum of these diagrams cancels.

Finally, we pick out the remaining ($\ln r$)-dependent parts of the $f_2(k = i\xi^{-1}; r)$ terms (considered in the appendix to order ϵ^2), insert them into Eq. (37), and obtain

$$\begin{aligned} f_2(k = i\xi^{-1}; r) &= 288[(1 - \tfrac{1}{2}\epsilon \ln r)\, C_2 - 72u_0 D_2(r)\, C_2] \\ &\approx 288(1 + \tfrac{1}{2}\epsilon \ln r)\, C_2 \end{aligned} \tag{42}$$

where $72u_0 D_2(r)\, C_2 \approx -\epsilon (\ln r)\, C_2$ is the sum of the ($\ln r$)-dependent parts of diagrams (d) and (e). Since the term $f_2(k = i\xi^{-1}; r)$ enters Eq. (34) multiplied by the factor $u_0^2\overline{m}^2 \sim \epsilon r(1 - \frac{1}{2}\epsilon \ln r)$, we find that the contribution of f_2 to the term $r \ln r$, up to second order in ϵ, is equal to zero.

Thus, again, $\rho = \eta$, and we arrive at the conclusion that Eq. (1) is satisfied both above and below T_c, at least to order ϵ^2.

7. SCALED EQUATION FOR THE CORRELATION FUNCTION

The homogeneity of the correlation function as a function of k and ξ follows from general considerations.[6,11,12]. We proceed to show explicitly this homogeneity in the framework of our calculation. The considerations are analogous to those of the previous section.

Taking into account all the terms contributing up to second order in ϵ, the inverse correlation function is calculated from Eqs. (5), (8), and (28) (the $n = 1$ case is considered here) as

$$g^{-1}(k; r) = Ak^{2-\eta} + r - u_0^2 f_1(k; r) - u_0^2 \overline{m}^2 f_2(k; r) \tag{43}$$

where f_1 and f_2 have been introduced in Eqs. (35) and (37). The value of the coefficient A is

$$A = 1 + (\epsilon^2/54)(\tfrac{5}{4} + \ln \Lambda)$$

In order to put all integrals into explicit dependence on the single scaled variable x, with

$$x = A^{1/(2-\eta)} k\xi \tag{44}$$

some simple transformations were performed (see the appendix).
To lowest order in ϵ,

$$f_1(k; r) = 96[\Delta D_3(k; r) - \Delta D_3(k; 0)] \approx 96 r \mathscr{D}_3(x) \tag{45}$$

[appendix, Eq. (A.3)]. The second term in Eq. (43), $f_2(k; r)$, is the sum of $\overline{m}$-dependent contributions. When each of them is calculated to the needed order in ϵ, it can be reduced to the form [appendix, Eqs. (A.1), (A.4), . . ., (A.7)]

$$\begin{aligned} f_2(k; r) = 288[(1 - \tfrac{1}{2}\epsilon \ln r)\mathscr{D}_2(x) - 72u_0 D_2(r)\mathscr{D}_2(x) - 12u_0\mathscr{D}_2^2(x) \\ - 48u_0\mathscr{D}_4(x) + 72u_0\mathscr{D}_4'(x) + 72u_0\mathscr{D}_5(x)] \end{aligned} \tag{46}$$

where, due to the renormalization of the self-energy, Eq. (8), the same cancellation of diagrams (b), (c), and (f) (see Fig. 5) at $\mathbf{q} = 0$ ($\mathbf{q}$ is the transferred momentum inside the "bubble") was taken into account. Furthermore, following exactly the procedure of Section 6, the ($\ln r$) dependence of $f_2(k; r)$ in Eq. (43) is cancelled up to order ϵ^2 by the factor $u_0^2\overline{m}^2 \sim \epsilon r(1 - \tfrac{1}{2}\epsilon \ln r)$.

Finally, introducing y via

$$y = g^{-1}(k; r)/g^{-1}(k; 0) = g^{-1}(k; r)/Ak^{2-\eta} \tag{47}$$

the scaled equation for the correlation function, to second order in ϵ, is obtained as

$$\begin{aligned} x^{2-\eta} y = 1 + x^{2-\eta} - 36u_0\mathscr{D}_2(x) - 48u_0^2[2\mathscr{D}_3(x) - 9\mathscr{D}_2^2(x) \\ - 36\mathscr{D}_4(x) + 54\mathscr{D}_4'(x) + 54\mathscr{D}_5(x)] \end{aligned} \tag{48}$$

APPENDIX

All Δ integrals to be considered here are convergent ones. This can be seen from their expressions, presented below. Therefore there is no need to introduce a momentum cutoff, as, for example, in $D_2(k = 0; r)$[see Eq. (20)]. Once we set $\Lambda \to \infty$ the explicit r dependence of these integrals can be extracted by the simple replacement of variables: $\mathbf{q} = r^{1/2}\mathbf{q}_1$, $\mathbf{p} = r^{1/2}\mathbf{q}_2$, and $\mathbf{k} = r^{1/2}\mathbf{n}x$ ($\mathbf{n}$ is a unit vector). We proceed as follows.

The contribution of the first-order diagram (a) (Fig. 5) to the term $\ln r$ has to be evaluated up to first order in ϵ; it reads

$$\Delta D_2(k; r) \approx (1 - \tfrac{1}{2}\epsilon \ln r)\, \mathscr{D}_2(x) \qquad \text{A.1}$$

where

$$\mathscr{D}_2(x) = \int d^{4-\epsilon}q_1(\mathbf{q}_1{}^2 + 1)^{-1}\{[(\mathbf{q}_1 + \mathbf{n}x)^2 + 1]^{-1} - (\mathbf{q}_1{}^2 + 1)^{-1}\}$$

In order to calculate the correlation length, all diagrams have to be evaluated at $\mathbf{k} = i\xi^{-1}\mathbf{n}$, i.e., $x = i$. The result for diagram (a) can be denoted as $\mathscr{D}_2(x = i) = C_2$, $C_2 = \text{const.}$

All remaining terms—contributions of the second-order diagrams [(b)–(f), Fig. 5]—to lowest order in ϵ ($\epsilon = 0$) can be given in the following form.

The common part of diagrams (b) and (c):

$$\Delta D_2'(k; r) = r^{-1}\mathscr{D}_2'(x) \qquad \text{(A.2)}$$

with

$$\mathscr{D}_2'(x) = \int d^4q_1(\mathbf{q}_1{}^2 + 1)^{-2}\{[(\mathbf{q}_1 + \mathbf{n}x)^2 + 1]^{-1} - (\mathbf{q}_1{}^2 + 1)^{-1}\}$$

The single diagram without order-parameter "legs" (Fig. 4) produces

$$\Delta D_3(k; r) - \Delta D_3(k; 0) = r\mathscr{D}_3(x)$$

$$\begin{aligned}\mathscr{D}_3(x) = \int & d^4q_1\, d^4q_2(\mathbf{q}_1{}^2 + 1)^{-1}(\mathbf{q}_2{}^2 + 1)^{-1} \\ & \times \{[(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{n}x)^2 + 1]^{-1} - [(\mathbf{q}_1 + \mathbf{q}_2)^2 + 1]^{-1} \\ & - (\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{n}x)^{-2} + (\mathbf{q}_1 + \mathbf{q}_2)^{-2}\}\end{aligned} \qquad \text{(A.3)}$$

The diagram (d) (Fig. 5) can be naturally divided into ($\ln r$)-dependent and -independent parts as

$$\begin{aligned}\Delta D_2{}^2(k; r) &= D_2{}^2(k; r) - D_2{}^2(0; r) = 2D_2(r)\,\Delta D_2(k; r) + [\Delta D_2(k; r)]^2 \\ &\approx 2D_2(r)\,\mathscr{D}_2(x) + \mathscr{D}_2{}^2(x)\end{aligned} \qquad \text{(A.4)}$$

In order to extract the ($\ln r$) dependence from diagrams (e) and (f) [Eqs. (39) and (40)], that is, to separate the value of the "bubble" at $\mathbf{q} = 0$ as the (ln r)-dependent part from the remaining (ln r)-independent part of the diagram, we use the following simple transformation:

$$[(\mathbf{q}+\mathbf{p})^2+r]^{-1}=(\mathbf{p}^2+r)^{-1}+\{[(\mathbf{q}+\mathbf{p})^2+r]^{-1}-(\mathbf{p}^2+r)^{-1}\}$$

The result is

$$\Delta D_4(k;r)=D_2(r)\,\mathscr{D}_2(x)+\mathscr{D}_4(x);\qquad \mathscr{D}_4(x=i)=C_4 \tag{A.5}$$

$$\Delta D_4'(k;r)=r^{-1}[D_2(r)\,\mathscr{D}_2'(x)+\mathscr{D}_4'(x)];\qquad \mathscr{D}_4'(x=i)=C_4' \tag{A.6}$$

where

$$\mathscr{D}_4(x)=\int d^4q_1\,d^4q_2(\mathbf{q}_1^{\,2}+1)^{-1}(\mathbf{q}_2^{\,2}+1)^{-1}\,F(\mathbf{q}_1\,,\mathbf{q}_2\,;x)$$

$$\mathscr{D}_4'(x)=\int d^4q_1\,d^4q_2(\mathbf{q}_1^{\,2}+1)^{-2}(\mathbf{q}_2^{\,2}+1)^{-1}\,F(\mathbf{q}_1\,,\mathbf{q}_2\,;x);$$

and

$$F(\mathbf{q}_1\,,\mathbf{q}_2\,;x)=\{[(\mathbf{q}_1+\mathbf{n}x)^2+1]^{-1}-(\mathbf{q}_1^{\,2}+1)^{-1}\}\{[(\mathbf{q}_1+\mathbf{q}_2)^2+1]^{-1}-(\mathbf{q}_2^{\,2}+1)^{-1}\}$$

Finally, diagram (g) does not contribute to the term ln r to second order in ϵ:

$$\Delta D_5(k;r)=r^{-1}\mathscr{D}_5(x) \tag{A.7}$$

$$\mathscr{D}_5(x)=\int d^4q_1\,d^4q_2(\mathbf{q}_1^{\,2}+1)^{-1}(\mathbf{q}_2^{\,2}+1)^{-1}[(\mathbf{q}_1-\mathbf{q}_2)^2+1]^{-1}\times\{[(\mathbf{q}_1+\mathbf{n}x)^2+1]^{-1}[(\mathbf{q}_2+\mathbf{n}x)^2+1]^{-1}-(\mathbf{q}_1^{\,2}+1)^{-1}(\mathbf{q}_2^{\,2}+1)^{-1}\}$$

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